Modelling musical instruments

Abstract

In this study, we examined and modelled the profiles of normal mode guitar and piano strings with respect to time, distance, and frequency. We analysed the behaviour of the strings under the seven lowest frequencies, and compared the normal mode frequencies to the expected ones, finding that the higher the frequency the more it is detuned, and this effect is even stronger for the piano string.

1 Introduction

Playing on a musical instrument is a common thing today. The most popular ones are probably the guitar and the piano. But does one understand how they actually work? Where does that sound of a particular key come from? Why do they sound differently from one another? And what does it mean for an instrument, to "be in tune"? In the following, we are going to discuss some of these questions by modelling the piano and the guitar string.

Human ears detect *longitudinal* vibrations, which can be created by, for example, vibrating strings at a given frequency. The higher the frequency(Hertz) of the vibration is, the higher sound we hear, thus we need to examine the frequency of the vibrating strings. Thinking about such string as a lot of masses connected by springs is a good approximation. Consider N masses(m_i), connected by springs with elastic coefficient k, between two fixed walls, such that the position of mass m_i is X_i:

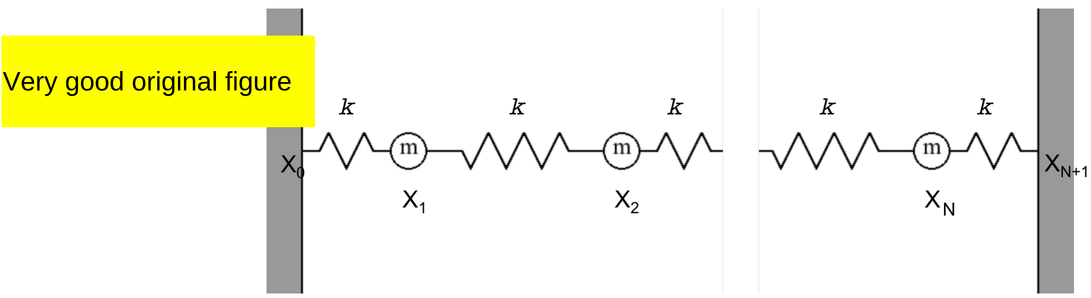

Figure 1: Representation of the string by the system of N masses with springs between them with spring constant k.

First we analyse the *longitudinal* behaviour of such system, meaning that the masses cannot move in the vertical direction, only along the x axis. The force acting on a spring is then given by $F = kx$, where x is the displacement of m from rest position. In our case, this displacement is equal to the horizontal difference between the adjacent nodes minus d, the rest length(without any mass connected to it) of the spring between them. Since we have two springs connected to one mass, each mass experience two counter forces. Therefore the total force on m_i is the difference of the 2 counter forces acting towards m_{i+1} and m_{i-1}:

$$F_i = k(X_{i+1} - X_i - d) - k(X_i - X_{i-1} - d) = k(X_{i+1} + X_{i-1} - 2X_i), \tag{1}$$

and by applying Newton's second law, the force acting on m_i at X_i is given by

$$m_i \frac{d^2 X_i}{dt^2} = k(X_{i+1} + X_{i-1} - 2X_i). \tag{2}$$

When there is no force acting on the system, $dX_i/dt = d^2X_i/dt^2 = 0$, the distance between the masses is

$$l_0 = \frac{X_{N+1} - X_0}{N+1}. \tag{3}$$

One can conclude that the distance to the ith mass is

$$X_i = X_0 + il_0 + y_i, \tag{4}$$

where y_i is the displacement of m_i from its static position. Combining (2) and (4) we can obtain

$$m_i \frac{d^2 y_i}{dt^2} = k(y_{i+1} + y_{i-1} - 2y_i), \tag{5}$$

where $i = 1, 2, 3, ..., N$ and $y_0 = y_{N+1} = 0$. One can rearrange this and write it up in matrix form as

$$\ddot{\mathbf{y}} = k\mathbf{M}^{-1}\mathbf{A}\mathbf{y}, \tag{6}$$

where $\mathbf{M} = \text{diag}(m_i)$, and

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & . & . & . & . & . \\ 1 & -2 & 1 & 0 & 0 & . & . & . & . & . \\ 0 & 1 & -2 & 1 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & 1 & -2 & 1 & 0 \\ . & . & . & . & . & 0 & 0 & 1 & -2 & 1 \\ . & . & . & . & . & 0 & 0 & 0 & 1 & -2 \end{pmatrix}. \tag{7}$$

Such system can be solved by seeking special solutions of the form

$$\mathbf{y}_\mu = \mathbf{v}_\mu sin(\omega_\mu t), \tag{8}$$

where $\mathbf{v}$ is the eigenvector and ω is the angular frequency for the corresponding mass and μ labels the unique solutions, up to N. Notice that $\mathbf{v}_\mu$ corresponds to the amplitude of the motion of the masses. Inserting (8) into (6) gives us the following eigenvalue problem

$$-\omega_\mu^2 \mathbf{v}_\mu = k\mathbf{M}^{-1}\mathbf{A}\mathbf{v}_\mu. \tag{9}$$

Solutions of (8) are called the normal mode solutions of (5). The frequency of the oscillations are given by

$$\nu_\mu = \frac{\omega_\mu}{2\pi}, \tag{10}$$

and writing $\mathbf{C}{=}k\mathbf{M}^{-1}\mathbf{A}$, we can rewrite (6) as

$$\ddot{\mathbf{y}} = \mathbf{C}\mathbf{y}. \tag{11}$$

2 Transverse vibrations

Q1

We now consider *transverse* vibration of the masses by fixing the horizontal coordinates and only allowing the vertical displacement, see Figure 2. One can compute the potential energy of such model for each mass. Assume that the fixed horizontal distance between the nodes is $l_0 = x_{i+1} - x_i$ and that the vertical displacement $y_{i+1} - y_i << l_0$. The general formula for potential energy is $V = \frac{1}{2}kD^2$, where D is the displacement of the compressed spring from its rest position. In this case, with the help of Figure 2 one can compute that the length of the compressed spring:

$$x_i = \sqrt{l_0^2 + (y_{i+1} - y_i)^2}. \tag{12}$$

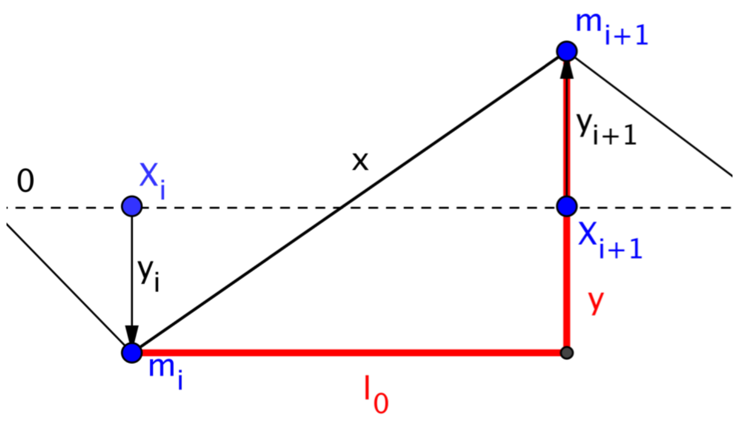

Figure 2: Modelling the transverse motion of the masses.

To get the displacement D, we subtract d, the rest length of the spring, from x:

$$D = \sqrt{l_0^2 + (y_{i+1} - y_i)^2} - d, \tag{13}$$

and hence the potential energy is

$$V_i = \frac{k}{2}\left(\sqrt{l_0^2 + (y_{i+1} - y_i)^2} - d\right)^2, \tag{14}$$

Since there are N masses, we can derive the expression for the potential energy of the whole system from (14):

$$V = \frac{k}{2}\sum_{i=0}^{N}\left(\sqrt{l_0^2 + (y_{i+1} - y_i)^2} - d\right)^2. \tag{15}$$

If we expand (15) we have

$$V = \frac{k}{2}\sum_{i=0}^{N}\left(l_0^2 + (y_{i+1} - y_i)^2 + d^2 - 2d(\sqrt{l_0^2 + (y_{i+1} - y_i)^2}\right). \tag{16}$$

✓

Introducing the new variables

$$a = l_0^2 \quad \text{and} \quad x = y_{i+1} - y_i, \tag{17}$$

one can compute the Taylor expansion of $\sqrt{a + x^2}$ and get

$$\sqrt{a + x^2} = \sqrt{a} + \frac{x^2}{2\sqrt{a}} - \dots \quad . \tag{18}$$

Note that since $x = y_{i+1} - y_i$ and $y_{i+1} - y_i << l_0$, we can neglect the terms which contains x^p for $p > 2$ in the numerator. Hence we have

$$\sqrt{l_0^2 + (y_{i+1} - y_i)^2} = l_0 + \frac{(y_{i+1} - y_i)^2}{2l_0}, \tag{19}$$

and (16) then simplifies to

$$V \approx \sum_{i=0}^{N}\left(\frac{k}{2}(l_0 + d^2 - 2dl_0) + \frac{k}{2}\left(1 - \frac{d}{l_0}\right)(y_{i+1} - y_i)^2\right). \tag{20}$$

Introducing the new variables

$$v_0 = \frac{k}{2}(l_0 + d^2 - 2dl_0) \quad \text{and} \quad \kappa = k\left(1 - \frac{d}{l_0}\right), \tag{21}$$

(20) becomes

$$V \approx \sum_{i=0}^{N} \left(v_0 + \frac{\kappa}{2}(y_{i+1} - y_i)^2 \right), \tag{22}$$

which is the quadratic approximation of the potential energy, and we have

$$m_i \frac{\partial^2 y_i}{\partial t^2} = -\frac{\partial V}{\partial y_i}. \tag{23}$$

We write out this sum as

$$V = Nv_o + \frac{\kappa}{2}\left[(y_1 - y_0)^2 + (y_2 - y_1)^2 + (y_3 - y_2)^2 + \dots\right]. \tag{24}$$

Notice that the potential energy at the end points is $V_0 = V_{N+1} = 0$, hence $F_0 = F_{N+1} = 0$, so to compute the force acting only on the masses without the end points, take now $i = 1$ and rewrite (24) as

$$V = Nv_o + \frac{\kappa}{2}\left[(y_i - y_{i-1})^2 + (y_{i+1} - y_i)^2 + (y_{i+2} - y_{i+1})^2 + \dots\right]. \tag{25}$$

Substituting (25) into (23) and simplifying that gives us

$$m_i \frac{\partial^2 y_i}{\partial t^2} = -\frac{\partial V}{\partial y_i} = \kappa(y_{i+1} + y_{i-1} - 2y_i). \tag{26}$$

Q2

Note that in (25) the derivatives, with respect to y_i, of the terms after $(y_{i+1} - y_i)^2$ are 0. By looking at Figure 2 and (21) one can conclude that l_0 cannot be smaller than d, and $l_0 = d$ only if there are no ~~masses~~ in the system and no force acting on it. So we have $d < l_0$ and therefore $\kappa > 0$.

stretching

3 Guitar string

The system discussed above, with very large number of masses, can be viewed as an approximately model for a guitar string. Assume that the end points are fixed, $X_0 = 0$ and $X_{N+1} = L$, and that the displacements y_i do not vary too much between nodes. Then we can use the Taylor series to approximate the terms y_{i+1} and y_{i-1}, which are identical to $y(x_i + l_0)$ and $y(x_i - l_0)$ respectively, in (26) or (5). Using this approximation and assuming that all the masses are equal, $m_i = m$, we have

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{27}$$

which is in fact the wave equation of the string, and $c = l_0\sqrt{\kappa/m}$. To solve this we need to apply the method of seperation of variables. Let $y = f(x)g(t)$, then (27) becomes Q3

$$\begin{aligned} f\frac{d^2 g}{dt^2} &= c^2 g \frac{d^2 f}{dx^2} \\ \frac{1}{g}\frac{d^2 g}{dt^2} &= c^2 \frac{1}{f}\frac{d^2 f}{dx^2} = \lambda, \end{aligned} \tag{28}$$

where λ is a constant since both sides of (28) are independent from each other: the left hand side depends on t while the right hand side depends on x. Then we have

$$\frac{\lambda}{c^2}f = \ddot{f} \quad \text{and} \quad \lambda g = \ddot{g} \tag{29}$$

which is a homogeneous second order linear constant coefficient ODE. The general solution for this ODE is

$$f(x) = Ae^{\sqrt{\lambda/c^2}x} + Be^{-\sqrt{\lambda/c^2}x}. \tag{30}$$

Given the fixed end points at $x = 0$ and $x = L$, we have the boundary conditions $f(0) = f(L) = 0$. Applying the first condition to (30) we have

$$\begin{aligned} f(0) = 0 = Ae^0 + Be^0 \\ A = -B \end{aligned} \tag{31}$$

and hence (30) with the second condition becomes

$$f(L) = 0 = A(e^{\sqrt{\lambda/c^2}L} - e^{-\sqrt{\lambda/c^2}L}), \tag{32}$$

which is 0 if and only if $A = 0$ or

$$\begin{aligned} e^{\sqrt{\lambda/c^2}L} &= e^{-\sqrt{\lambda/c^2}L} \\ e^{2L\sqrt{\lambda/c^2}} &= 1. \end{aligned} \tag{33}$$

We know that $e^{i\theta} = cos\theta + isin\theta$. In our case $i\theta = 2L\sqrt{\lambda/c^2}$, $cos(\theta) = 1$, and $sin(\theta) = 0$, therefore $\theta = 2n\pi$ for $n \in \mathbb{Z}$, and

$$\begin{aligned} i\theta = i2n\pi &= 2L\sqrt{\lambda/c^2} \\ \lambda &= -\left(\frac{cn\pi}{L}\right)^2 \end{aligned} \tag{34}$$

One can now rewrite (30) using (31) and (34):

$$f_n(x) = A(e^{\frac{n\pi}{L}ix} - e^{-\frac{n\pi}{L}ix}). \tag{35}$$

But with

$$\begin{aligned} e^{(\frac{n\pi}{L}ix)} &= cos\left(\frac{n\pi}{L}x\right) + isin\left(\frac{n\pi}{L}x\right), \\ e^{-(\frac{n\pi}{L}ix)} &= cos\left(-\frac{n\pi}{L}x\right) + isin\left(-\frac{n\pi}{L}x\right), \end{aligned} \tag{36}$$

and with basic trigonometric identities, (35) simplifies to

$$f_n(x) = 2Aisin\left(\frac{n\pi}{L}x\right). \tag{37}$$

Applying the same method, one can derive the most general solution for $g_n(t)$, that is

$$g_n(t) = \tilde{A}cos\left(\frac{n\pi c}{L}t\right) + \tilde{B}sin\left(\frac{n\pi c}{L}t\right), \tag{38}$$

where $\tilde{A}$ and $\tilde{B}$ are constants, and thus we have

$$y(x,t) = \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi}{L}x\right)\left(\tilde{A}_n cos(\frac{n\pi c}{L}t) + \tilde{B}_n sin(\frac{n\pi c}{L}t)\right). \tag{39}$$

Given (34) and the relation between the eigenvalue, λ_μ and the angular frequency, ω_μ,

$$\lambda_\mu = -\omega_\mu^2, \tag{40}$$

we conclude that

$$\omega_n = \frac{cn\pi}{L}, \tag{41}$$

and the frequency of the oscillations is

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{cn}{2L}. \tag{42}$$

Rearranging (41) and combining it with (37) we can derive an expression for $f_n(x)$ in terms of ω_n:

Take A = -i a to get a real solution

$$f_n(x) = 2Aisin\left(\frac{\omega_n}{c}x\right). \tag{43}$$

Above we solved the wave equation (27) for constant c, but when c depends on x, we need another approach. For now we stick with constant c. (27) is also an initial value problem, which can be solved by considering the initial values for y and $\dot{y}$ and (26) as a system $2N$ first order ODE: **Q4**

$$\begin{aligned} g_i(t) &= \frac{dy_i}{dt}, \\ \frac{dg_i(t)}{dt} &= \frac{\kappa}{m_i}(y_{i+1} + y_{i-1} - 2y_i). \end{aligned} \tag{44}$$

Figure 3 models the guitar string equipped with particular initial values at $t = 0$: $y(x) = sin(\pi x/L)$ and $\dot{y}(x) = 0$ for the left graph, $y(x) = sin(9\pi x/L)$ and $\dot{y}(x) = 0$ for the right one, and T is the period. From (37) it follows that $n = 1$ in the first case, which corresponds to the first normal mode, as shown on the graph, and $n = 9$ for the second case. Using (41) one can compute the frequencies of these profiles. **Q5**

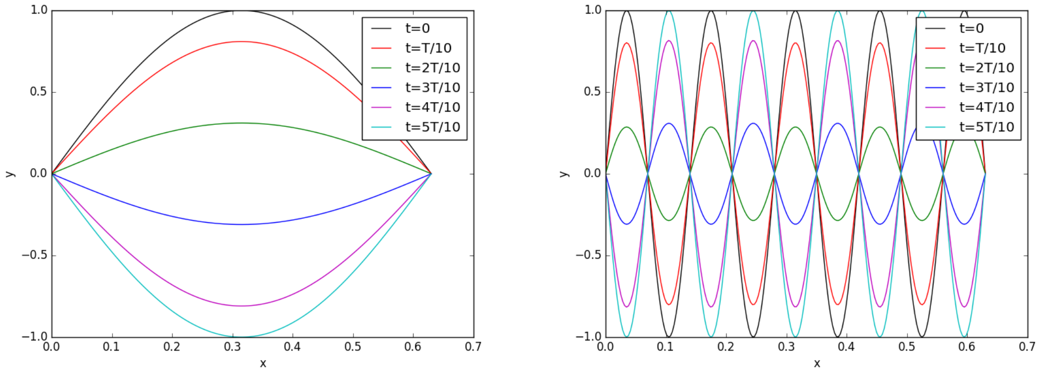

Figure 3: The profile of the string at 6 given times with the initial value $y(x) = sin(\pi x/L)$ on the left and $y(x) = sin(9\pi x/L)$ on the right.

We observe that as t increases, the string is travelling from the initial position(at the amplitude) to its first amplitude, which happens at $t = 5T/10$, which is clearly, the half of the period. Meanwhile, Figure 4 models the profile of the fixed point, $x = L/3$, on the string as a function of time. One can see that there are 20 peaks on the graph, meaning that this one point on the string reached the same height 20 times, so the system was running till $t = 20T$. Furthermore, on Figure 5 we plot the averaged modulus of the Fourier Transform($\sqrt{a_i^2 + b_i^2}$) of the vibrating string.

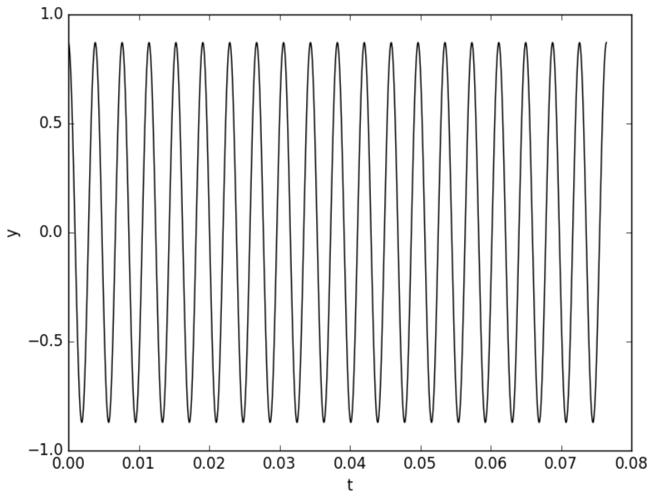

Figure 4: The behaviour of the string with the initial condition $y(x) = sin(\pi x/L)$ and $\dot{y}(x) = 0$ at $x = L/3$ as a function of time.

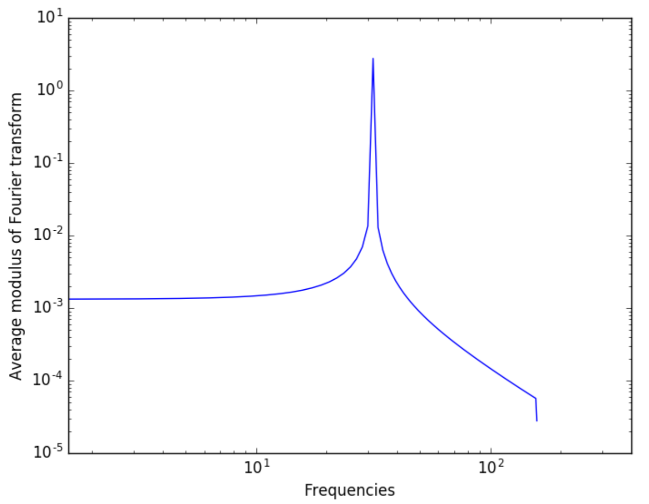

Figure 5: The logarithmic plot of the averaged modulus of the Fourier Transform of the vibrating string with the same initial conditions as above.

3.1 Normal modes at different frequencies

Q7

In the previous section we examined the behaviour of guitar string at various times but with a given initial condition, hence given frequency. This time we study the system at various frequencies, namely, at the lowest frequencies. The frequencies are obtained by (40), but we plot the eigenvector, $\mathbf{v}_\mu$, corresponding to the eigenvalue, λ_μ , since they represent the largest displacement of the mass m_μ based on (8)(see Figure 6).
The left hand side of Figure 7 shows the ratio ν_μ/ν_0 (in blue) and the identity function(in red) on as a function of the index number μ, starting from $\mu = 1$, and for $N = 200$. On the other hand, the one on the right shows ν_μ, the computed frequencies of the normal mode, as a function of the normal mode index μ(in blue), as well as the expected values of the frequencies(in red) based on (42), for $N = 900$.

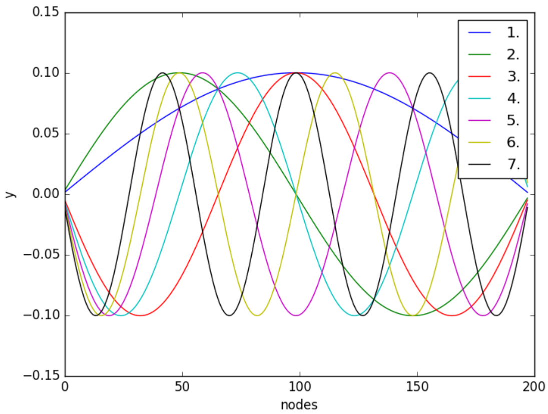

Figure 6: Normal modes of the string at the 7 lowest frequencies.

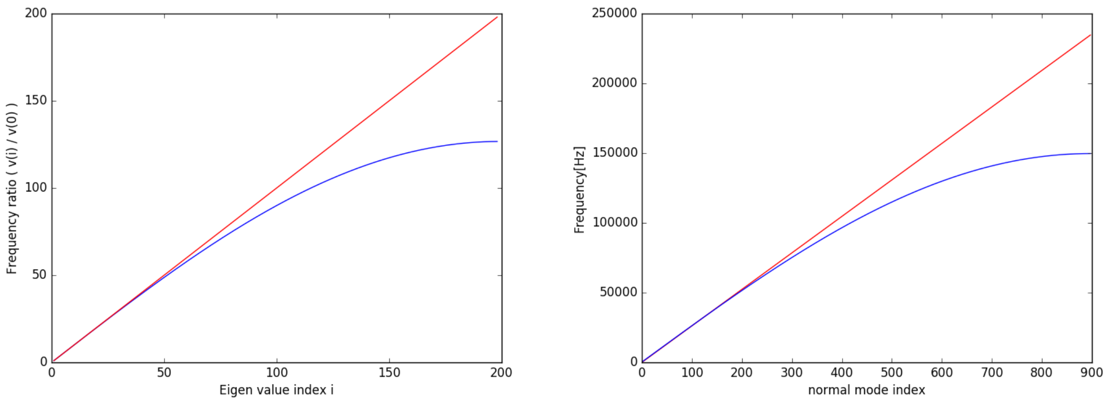

Figure 7: The ratio of the frequencies with the identity function on the left, and the computed and expected frequencies of the normal modes against their index on the right.

Q6

In the second plot of Figure 7 we used $N = 900$ because the smaller the N, the greater the difference is between the expected and the normal mode frequencies. For $N = 900$ the difference between ν_μ and $\mu\nu_0$ for $\mu < 100$ is smaller than 10 cents[1], hence it is a sensible value and gives a better approximation to the theoretical equation. For $N = 900$ the lowest frequency of the normal modes is $\nu_0 = 261.5872$ which is almost identical to the analytical solution, $\nu_0 = 261.5873$ based on (43), while it is $\nu_0 = 261.5845$ for $N = 200$.

3.2 Closer to reality

In real, one could hardly demonstrate the case discussed above. Instead, the guitar string is normally plucked by the player's finger. Then the string forms a triangular shape, and then it is released suddenly. We are now going to simulate the motion of this case, with the same parameters as before. One can notice the difference between the two graphs on both Figure 8 and Figure 9: the first ones are sharper, more stable than the second ones. This is due the greater number of nodes and hence masses used on the string. The more nodes we use, the better approximation we get for a real string. On the second graph of Figure 8 one can compare the profile at $t = 0$ and $t = 5T/10$: due to the small number of masses, clearly, the string does not reach the amplitude, $y = 1$, by the time it travels half a period. Furthermore, it becomes **Q8** wobbly by that time. We can conclude that modelling the plucked string is more tricky than the simple normal mode one, since there are more parameters affecting the system.

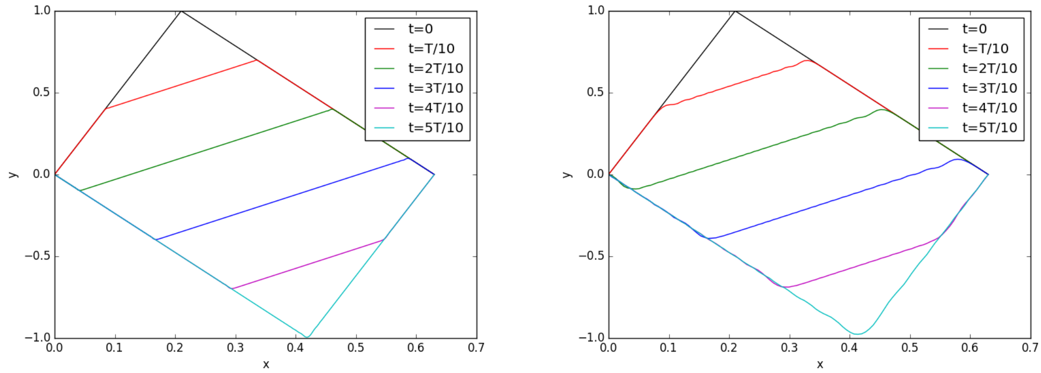

Figure 8: The profile of the first normal mode of the plucked guitar string(at $x = L/3$) at 6 different times for $N = 900$ on the left and $N = 100$ on the right.

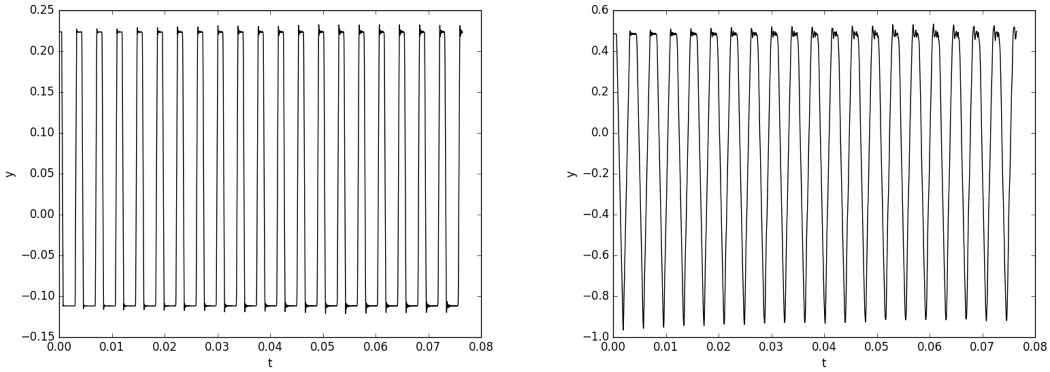

Figure 9: The motion of the point $x = L/3$ on the string for the first normal mode for $N = 900$ on the left and $N = 100$ on the right, as a function of time.

4 Piano string

4.1 Formula derivation

Q9

Piano strings are thicker than guitar strings and made out of metal, which requires more energy to bend them. This results in a slightly different equation from (27):

$$\frac{\partial^2 f}{\partial t^2} = c_1^2 \frac{\partial^2 f}{\partial x^2} - c_2^2 \frac{\partial^4 f}{\partial x^4}. \tag{45}$$

To derive the finite order approximation of the $\partial^4 f/\partial x^4$ term using the 5 points $x_i, x_{i\pm1}, x_{i\pm2}$, we apply the Taylor expansion of $f(x \pm dx)$ and $f(x \pm 2dx)$ up to the 5th order. We obtain

$$\begin{aligned} f_{i\pm1} = f(x_i \pm dx) =& f(x_i) \pm \frac{df(x_i)}{dx}dx + \frac{d^2 f(x_i)}{dx^2}\frac{dx^2}{2!} \pm \frac{d^3 f(x_i)}{dx^3}\frac{dx^3}{3!} \\ &+ \frac{d^4 f(x_i)}{dx^4}\frac{dx^4}{4!} \pm \frac{d^5 f(x_i)}{dx^5}\frac{dx^5}{5!} + o(dx^6), \\ f_{i\pm2} = f(x_i \pm 2dx) =& f(x_i) \pm \frac{df(x_i)}{dx}2dx + \frac{d^2 f(x_i)}{dx^2}\frac{4dx^2}{2!} \pm \frac{d^3 f(x_i)}{dx^3}\frac{8dx^3}{3!} \\ &+ \frac{d^4 f(x_i)}{dx^4}\frac{16dx^4}{4!} \pm \frac{d^5 f(x_i)}{dx^5}\frac{32dx^5}{5!} + o(dx^6). \end{aligned} \tag{46}$$

We need to take the following linear combination of the 4 equation in (46) to eliminate the derivative terms other than $\partial^4 f/\partial x^4$:

$$(f_{i+2} + f_{i-2}) - 4(f_{i+1} + f_{i-1}) = -6f_i + \frac{\partial^4 f}{\partial x^4}dx^4 + O(dx^6), \tag{47}$$

and rearranging this gives us

$$\Delta_i^4 = \frac{\partial^4 f}{\partial x^4} = \frac{f_{i+2} + f_{i-2} - 4f_{i+1} - 4f_{i-1} + 6f_i}{dx^4} + O(dx^2), \tag{48}$$

and hence (45) can be written as

$$\frac{\partial^2 f}{\partial t^2} = c_1^2 \Delta_i^2 f - c_2^2 \Delta_i^4 f. \tag{49}$$

The matrix form of (49) as follows

$$\frac{\partial^2 \mathbf{f}}{\partial t^2} = \frac{c_1^2}{dx^2}(A_2 + B_2)\mathbf{f} - \frac{c_2^2}{dx^4}(A_4 + B_4)\mathbf{f}, \tag{50}$$

where B_2 and B_4 are the given matrices for the boundary terms, A_2 is identical to (7) and A_4 is the matrix representation of the coefficients in (48). For instance A_4 for $N = 11$ is a 7×7 matrix, because there are 2 endpoints and 2 boundary conditions on the string: **Q10**

$$\mathbf{A}_4 \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}. \tag{51}$$

Notice that (50) can be rewritten in the form of (29) and hence it becomes an eigenvalue problem such as earlier. **Q11**

what are the expression for B2 and B4?

4.2 Modelling

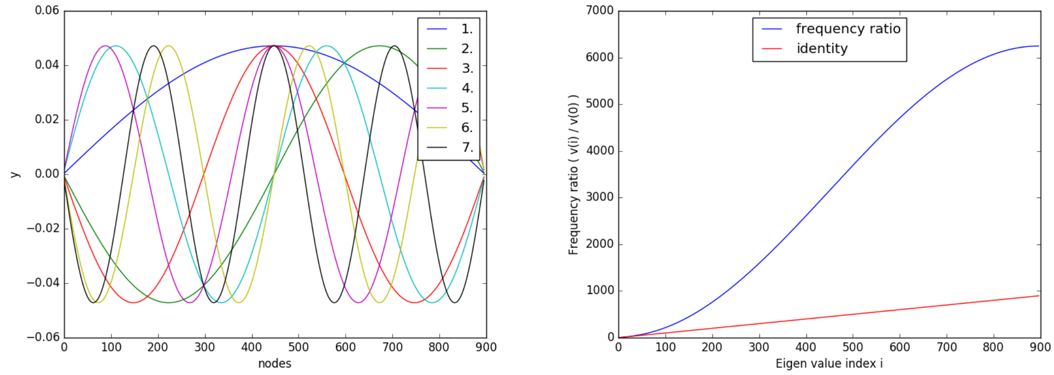

Figure 10: The motion of the piano string for the 7 lowest frequencies(left) and the frequency ratio ν_i/ν_0 against the index i with the identity function in red.

The behaviour of the piano string is shown on Figure 10. The shape of the profiles of the 7 lowest frequencies are similar to the guitar string profiles in Figure 6, but second graph is different from Figure 7, meaning that the frequency is changing differently as we go down on the piano string. This could be due to the fact that in this case, the string has additional 2 boundary conditions apart from the 2 endpoints, and that it is made of metal.

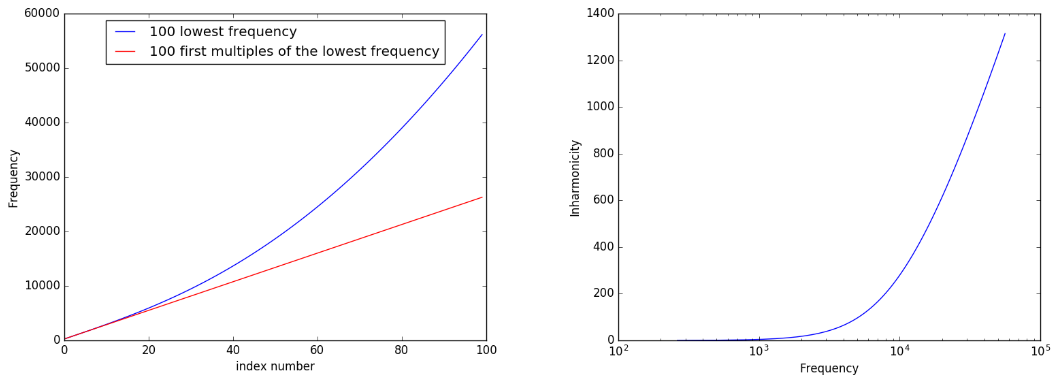

Figure 11: The inharmonicity against the frequency on the left, and the first 100 lowest calculated frequency and its expected values.

Figure 11 shows as well that the further we are on the string, the greater the difference is between the expected and the normal mode frequency. In addition, the graph on the left tells us that the inharmonicity(the discrepancy from the expected frequency) exponentially increases with increasing normal mode frequency. The 19th frequency, $\nu_{19} = 5310.08808835$ Hz, is the first frequency which is already detuned by over 1 semi tone(100 cents). All in all, can conclude that the piano string is harder to tune compared to the guitar string.

References

[1] http://hyperphysics.phy-astr.gsu.edu/hbase/Music/cents.html