2018-19 Optimisation–Summative

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Deadline: 2019-02-22. Submission: DUO.

1 Background

Let $G = (V, E)$ be a simple graph on n vertices. The neighbourhood of a vertex is $N(v) = \{u \in V : uv \in E\}$. A clique in G is a set of vertices $K \subseteq V$ such that for any distinct $u, v \in K$, $uv \in E$. In particular, the empty set is a clique; every singleton $\{v\}$ is a clique; and $\{u, v\}$ is a clique if and only if $uv \in E$. Let $\mathcal{K}(G)$ denote the collection of all cliques of G.

For any subset of vertices $S \subseteq V$, we introduce the variable x_S. Consider the following two LP problems based on those variables.

First, the **fractional clique cover number** $\pi^*(G)$ is the optimal value of:

$$\begin{aligned} \text{Minimise} \quad & \sum_{S \subseteq V} x_S \\ \text{s.t.} \quad & \sum_{S : v \in S} x_S \geq 1 \quad \forall v \in V \\ & x_S = 0 \quad \forall S \notin \mathcal{K}(G) \\ & x_S \geq 0 \quad \forall S \subseteq V. \end{aligned}$$

Note that the objective function is the sum over all subsets S of V.

Second, the **Shannon entropy** $\eta(G)$ is the optimal value of:

$$\begin{aligned} \text{Maximise} \quad & x_V \\ \text{s.t.} \quad & x_\emptyset = 0 \\ & x_{\{v\}} \leq 1 \quad \forall v \in V \\ & x_{N(v)\cup\{v\}} - x_{N(v)} = 0 \quad \forall v \in V \\ & x_T - x_S \geq 0 \quad \forall S \subseteq T \subseteq V \\ & x_S + x_T - x_{S\cup T} - x_{S\cap T} \geq 0 \quad \forall S, T \subseteq V \\ & x_S \geq 0 \quad \forall S \subseteq V. \end{aligned}$$

For instance, for the complete graph K_n, we have $\pi^*(K_n) = 1$ and $\eta(K_n) = n - 1$. In general, for any G on n vertices,

$$\eta(G) + \pi^*(G) \geq n.$$

More information on both LPs can be found in [1] and references therein.

2 Tasks

The aim of this assignment is to write a program that given a graph G, computes $\pi^*(G)$ and $\eta(G)$. You may use any solver that is either included in the MDS distribution or freely available to solve the LP. For extra marks (see below), you need to find ways to modify the LP into an equivalent one (e.g. remove some redundant constraints, change the objective function, etc.) in order to make it either easier to handle or to speed-up solving it. For both LPs, the optimal solution $\bar{x}$ is a vector of rational numbers (and its value is rational too). Therefore, for extra marks, both $\bar{x}$ and its value should be returned in rational format.

You need to submit your **code** and a short **report** (4 pages maximum). The report should contain:

- instructions on how to use your program;
- how you modified the LPs in order to manipulate them easily or speed-up solving them;
- an example of a graph G on eight vertices and the corresponding solutions and values for $\pi^*(G)$ and $\eta(G)$;
- and anything else you think I should know.

3 Marking scheme

For each LP, here are the following requirements:

1. **(20 marks)** Returns a correct solution with the right optimal value.
2. **(5 marks)** Handles graphs of up to eight vertices.
3. **(10 marks)** LP is modified for handling/speed-up.
4. **(10 marks)** Returns the solution and its value in rational format.
5. **(5 marks)** Is easy to use (easy to input/load a graph and to save the output).

References

[1] Maximilien Gadouleau. On the possible values of the entropy of undirected graphs. *Journal of Graph Theory*, 82:302–311, 2018.